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THE PROBLEM OF "GLOBAL COLOR" IN GAUGE THEORIESP.A.Horváthy[§]J.H.Rawnsley[§]

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ABSTRACT

The problem of "global color" (which arose recently in monopole theory) is generalized to arbitrary gauge theories: a subgroup K of the "unbroken" gauge group G is implementable iff the gauge bundle reduces to the centralizer of K in G . Equivalent implementations correspond to equivalent reductions. Such an action is an internal symmetry for a given configuration iff the Yang-Mills field reduces also. The case of monopoles is worked out in detail.

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1. INTRODUCTION

One of the most exciting problems which arose recently in monopole theory is that of global color [1-7]. We formulate it in two steps [7]: first, we would like to define the action of a fixed element g of the (unbroken) gauge group. Under usual conditions this presents no problem. In topologically non-trivial situations, however, this may not be possible. This is the problem of implementability. Next, if we are able to define such an action, when do we get a symmetry (in the sense of Schwarz [8] and Forgács and Manton [9]) for a given field configuration? The importance of these notions is seen, for example, from their role played in deriving conserved charges in gauge theories [7,35]. In this paper we give the mathematical solution to these problems. Our theory (formulated in fibre-bundle terms [8,10-12]) is valid for any classical gauge theory. Notice that the problem studied here is a special case of dimensional reduction [9,15].

Our starting point is Prop. 2.2, which states that a "rigid" internal action of a subgroup K of G on P exists if and only if P reduces to an $H = Z_G(K)$ - bundle Q . Furthermore, there is a (1-1) correspondence between such equivalent actions and isomorphic reductions (Props. 2.4 - 2.5).

An action of K on the principal bundle P induces an action of K also on the YM field. Similarly, we can study the action of K on matter fields - sections of bundles associated to P . The condition for such an action to exist is expressed again in terms of bundle-reduction (Theorem 3.1).

When is an action a symmetry for a given field configuration? Prop. 4.3 tells us, that the action of K on (P, G) defined by (Q, H) is an internal symmetry for a Yang-Mills connection A if and only if A reduces to a connection on Q . This happens if and only if H contains the holonomy group of A . The implementation of an internal symmetry-subgroup is necessarily unique. There is an analogous statement (Prop. 4.4) for matter fields.

These theorems provide us with a complete solution of the color problem - when we are able to construct a the

corresponding reductions. A first illustration is given by the non-Abelian Bohm-Aharonov experiment of Wu and Yang [33-35], where $G = SU(2)$ admits two inequivalent implementations.

The principal application of our theory is to non-Abelian monopoles [1-7]. Their basic properties [17-23] are geometrically reformulated in Section 5. The reduction of monopole bundles is worked out in Section 6. (As a by-product, we obtain also the topological theory of the "fate" of Grand Unified monopoles under successive symmetry breakings [24-28]).

The results are summarized as follows: denote by G the residual symmetry group of a monopole having $[P] \in \pi_1(G)$ as fundamental topological invariant. A subgroup K of G is implementable iff $[P]$ belongs to the image of $i_*: \pi_1(Z_G(K)) \rightarrow \pi_1(G)$ induced by the inclusion $i: Z_G(K) \rightarrow G$. Furthermore, the inequivalent implementations are labelled by the elements of $\pi_2(G/Z_G(K))$. In particular, the implementation of the full G is unique (when it does exist).

These results are conveniently expressed in terms of the "non-Abelian charge" Π of Goddard, Nuyts and Olive [17]: let us decompose Π as $\Pi = z(\Pi) + \Pi'$, where $z(\Pi) \in Z(\mathfrak{g})$ and $\Pi' \in [\mathfrak{g}, \mathfrak{g}]$. We prove that G is implementable iff either (i) $[\exp i\pi t \Pi] \in \pi_1(G)_{\text{free}}$, and $z(\Pi)$, the projection of the non-Abelian charge onto the centre, is quantized: $\exp i\pi z(\Pi) = 1$. Equivalently, iff (ii) $\exp i\pi t \Pi'$, $0 \leq t \leq 1$, is a contractible loop. G is a symmetry for a monopole given by Π iff $\Pi \in Z(\mathfrak{g})$.

The general results are illustrated on $SO(3)$ monopoles [18,17].

2. INTERNAL ACTIONS ON PRINCIPAL BUNDLES

Let P be a right principal G -bundle over a connected manifold M . A subgroup K of G acts internally on P , if we are given a left action $p \mapsto k \cdot p$ of K on P , which preserves each fibre and commutes with the (right) action of G , $k \cdot (pg) = (k \cdot p)g$, $\forall k \in K, g \in G, p \in P$, cf. [1-7]. If so, define the map $\tau_p : K \rightarrow G$ by $k \cdot p = p(\tau_p(k))$. τ_p is well-defined, since $k \cdot p$ belongs to the same fibre as p , and G acts on each fibre transitively and freely. $k \mapsto \tau_p(k)$ is a homomorphism of K into G , which satisfies $\tau_{pg} = \text{Ad}g^{-1} \circ \tau_p$, $g \in G$. In what follows we consider only the case when K acts on P freely; i.e. the homomorphism $\tau_p : K \rightarrow G$ is injective for each p . This can always be assumed without loss of generality for symmetries (see Section 4).

Choosing a local section $s_\alpha : V_\alpha \rightarrow P$, τ is given by $\tau^\alpha : V_\alpha \rightarrow \text{Hom}(K, G)$, where $\tau^\alpha_x = \tau_{s_\alpha(x)}$. If $h_{\alpha\beta} : V_\alpha \cap V_\beta \rightarrow G$ denotes the transition function of P , then $\tau^\alpha_x = \text{Ad}h_{\alpha\beta} \tau^\beta_x$, $x \in V_\alpha \cap V_\beta$.

An internal action of K on P is called rigid, if there exists a local trivialization $\{V_\alpha, s_\alpha\}$ of P such that each τ^α is constant [1-7]. If so, there is no loss of generality in assuming that $\tau^\alpha_x(k) = k$ for each x . In such a gauge

$$(2.1) \quad k \cdot s_\alpha(x) = s_\alpha(x)k, \quad \forall x \in V_\alpha, k \in K.$$

PROPOSITION 2.1

An internal action of K on P is rigid if and only if the image of the associated map $\tau : P \rightarrow \text{Hom}(K, G)$ is the orbit of the inclusion map $i : K \hookrightarrow G$ under the adjoint action of G on $\text{Hom}(K, G)$.

Proof: Suppose the action of K on P is rigid in a gauge $\{V_\alpha, s_\alpha\}$. If $p \in P$ is such that $\pi(p) \in V_\alpha$, where π is the projection $\pi : P \rightarrow M$, then $p = s_\alpha(\pi(p))g$ for some $g \in G$. By (2.1)

$$\begin{aligned} p\tau_p(k) &= k \cdot p = k \cdot (s_\alpha(\pi(p))g) = (k \cdot s_\alpha(\pi(p)))g = \\ &= s_\alpha(\pi(p)k)g = p(g^{-1}kg). \end{aligned}$$

Hence $\tau_p(k) = \text{Ad}g^{-1}k$ and so $\tau_p = \text{Ad}g^{-1}i$, and thus the image is the orbit of i . Conversely, if τ has a single orbit as its image, we can always choose local gauges s_α so that τ^α is

constant, equal to a base point, which in this case is the inclusion map $i : K \hookrightarrow G$. In this gauge the action of K is rigid.

Requiring rigidity is seen easily to be the same as to require that, for each p , τ_p is the restriction to K of an automorphism of G [1,2,6,7].

Let us now consider a rigid internal action of $K \subset G$ and let H denote the stabilizer of $i : K \hookrightarrow G$ under the adjoint action of G ,

$$(2.2) \quad H = Z_G(K) = \{g \in G \mid \text{Ad}gk = k, \forall k \in K\}.$$

The orbit of i is identified with G/H , and τ can be viewed as a section of the associated bundle with fibre G/H . Any such section defines a reduction of P to an H -bundle. The reductions to H bundles are in known to be in (1-1) correspondence with sections of the associated bundle $P \times_G (G/H) \cong P/H$, and so with rigid actions of K on P . Hence we have proved

PROPOSITION 2.2

A rigid internal action of a subgroup K of G on P exists if and only if P reduces to $H = Z_G(K)$.

It is easy to see directly that the existence of a rigid action forces the bundle to reduce by using the special gauge (2.1), for, if $s_\beta = s_\alpha h_{\alpha\beta}$, then

$$s_\alpha(kh_{\alpha\beta}) = k \cdot s_\alpha h_{\alpha\beta} = k \cdot s_\beta = s_\beta k = s_\alpha h_{\alpha\beta} k$$

showing that $h_{\alpha\beta}$ commutes with K , and hence (P, G) reduces to an H -bundle Q .

COROLLARY 2.3

G itself acts on P internally and rigidly if and only if (P, G) reduces to $Z(G)$, the centre of G . In this case the transition functions take their values in $Z(G)$ in a suitable gauge.

Now we turn to the question of uniqueness of the action of K . To be able to discuss several actions at once we use the following notation: we denote by $\mu : K \times P \rightarrow P$ the map

$\mu(k, p) = k \cdot p$. Two actions given by μ_1 and μ_2 should be regarded as equivalent if they differ only by a gauge transformation, i.e. if there exists a bundle automorphism $\sigma: P \rightarrow P$ preserving fibres, commuting with the G -action, and such that $\mu_2(k, \sigma(p)) = \sigma(\mu_1(k, p)) \forall k \in K, p \in P$.

Equivalence preserves rigidity, since if $s_\alpha: V_\alpha \rightarrow P$ is a gauge in which μ_1 is constant then $\sigma \circ s_\alpha$ is a gauge in which μ_2 is constant.

An equivalence σ determines a map $\gamma: P \rightarrow G$ by

$$(2.3) \quad \sigma(p) = p\gamma(p),$$

which satisfies $\gamma(pg) = \text{Adg}^{-1}\gamma(p)$, so γ is a section of the bundle associated to P with G acting on itself by internal automorphisms. If we have two actions μ_1, μ_2 of K with corresponding maps $\tau^1, \tau^2: \text{Hom}(K, G)$ then

$$\sigma(p)\tau_{\sigma(p)}^2(k) = \mu_2(k, \sigma(p)) = \sigma(\mu_1(k, p)) = \sigma(p)\tau_p^1(k)$$

so

$$(2.4) \quad \tau_p^1(k) = \tau_{\sigma(p)}^2(k) = \text{Ad}\gamma(p)^{-1}\tau_p^2(k).$$

Conversely, given a section γ of $P \times_G G$ satisfying (2.4) then (2.3) defines an equivalence of the two actions.

In the rigid case we may assume $i: K \hookrightarrow G$ is a subgroup and the actions of K correspond with reductions Q of P to $H = Z_G(K)$ -bundles. Here τ has values in the G -orbit of i and $Q = \{p \in P \mid \tau_p = i\}$.

If we have two rigid actions μ_1 and μ_2 with corresponding reductions Q_1 and Q_2 then an equivalence σ of μ_1 and μ_2 implies $\tau_p^1 = \tau_{\sigma(p)}^2$ by (2.4), so $Q_2 = \sigma(Q_1)$. This suggests a notion of equivalence of reductions: two reductions Q_1 and Q_2 of a principal G -bundle P are equivalent if there is an automorphism σ of P preserving fibres with $Q_2 = \sigma(Q_1)$. Then we have

PROPOSITION 2.4

Two rigid actions μ_1, μ_2 are equivalent if and only if their corresponding reductions Q_1 and Q_2 are equivalent.

Obviously, if two reductions Q_1, Q_2 of P are equivalent as

reductions then they are isomorphic as H -bundles. This is in fact the only condition to be satisfied as the next result shows:

PROPOSITION 2.5

There is a (1-1) correspondence between isomorphisms of Q_1 and Q_2 as H -bundles and equivalences of Q_1 and Q_2 as reductions of P .

Proof: It remains only to show how to extend an isomorphism $\sigma_0: Q_1 \rightarrow Q_2$ to an isomorphism of P . For any $p \in P$ we find $g \in G$ (not unique) with $pg \in Q_1$ (since Q_1 is a reduction of P). Any other choice of g has the form gh with $h \in H$, but

$$\sigma_0(pg)g^{-1} = \sigma_0(pgh)(gh)^{-1},$$

since σ_0 is an H -map, so $\sigma(p) = \sigma_0(pg)g^{-1}$ gives a well-defined map $\sigma: P \rightarrow P$ with $\sigma(Q_1) = \sigma_0(Q_1) = Q_2$. It is easy to check σ is an automorphism of P .

3. INTERNAL ACTIONS ON ASSOCIATED BUNDLES

A matter field Φ is specified by giving a unitary representation $U: G \rightarrow U(E)$ (the set of unitary transformations of a linear space E), and by selecting, in each V_α , a local representative $\Phi^\alpha: V_\alpha \rightarrow E$ such that

$$\Phi^\alpha(x) = U(h_{\alpha\beta}(x))\Phi^\beta(x), \quad x \in V_\alpha \cap V_\beta.$$

Suppose a subgroup K of G acts on Φ pointwise and linearly. This means that $k \in K$ sends Φ to an object we denote by $(k \cdot \Phi)$, expressed locally as

$$(3.1) \quad (k \cdot \Phi)^\alpha(x) = U^\alpha_x(k)\Phi^\alpha(x)$$

where each U^α_x is a representation of K on E . For (3.1) to be well-defined we need the consistency condition

$$(3.2) \quad U^\alpha_x(k) = U[h_{\alpha\beta}(x)]U^\beta_x(k)U[h_{\alpha\beta}^{-1}(x)].$$

Φ is a section of the associated bundle $\mathcal{A} = P \times_G E$. (3.2) requires therefore that the $U^\alpha_x(k)$'s piece together to give sections of the bundle associated to P with fibre $U(E)$, where G acts by conjugation by $U(g)$. If the action of K on the fibre at x is denoted by $U_x(k)$, the $U^\alpha_x(k)$'s are local representatives of this action. $U(k)$ is a section of $U(\mathcal{A})$.

If the representation U is not faithful, denote by N its kernel. N is a normal subgroup of G , and $G^* = G/N$ is a group to which U descends to give a faithful representation U^* . $P \times_G E$ is naturally isomorphic to $P^* \times_{G^*} E$, where $P^* = P/N = P \times_G (G/N)$ is the principal G^* -bundle associated to the homomorphism $G \rightarrow G^*$. In this way we may reduce to the case where U^* is faithful but note that now K need not be a subgroup of G^* . This defect can be avoided if we assume that the action of K is induced locally by gauge transformations, i.e., if we assume that, for each k , there exist functions $k_\alpha: V_\alpha \rightarrow G$ such that $U^\alpha_x(k) = U(k_\alpha(x))$. These k_α must satisfy $U[k_\alpha(x)] = U[h_{\alpha\beta}(x)k_\beta(x)h_{\alpha\beta}^{-1}(x)]$, so

$$(3.3) \quad k_\alpha(x)^{-1}h_{\alpha\beta}(x)k_\beta(x)h_{\alpha\beta}^{-1}(x) \in N.$$

If we denote by g^* the projection of $g \in G$ into G^* , then

$$(3.4) \quad k^*_\alpha(x) = h^*_{\alpha\beta}(x)k^*_\beta(x)h^*_{\alpha\beta}(x)^{-1}.$$

It follows that K^* defines a section of $P^* \times_{G^*} [G^*]$ with G^* acting by the adjoint representation on itself. Since each U^α_x is a homomorphism, then, although $k \mapsto k_\alpha(x)$ need not be a homomorphism, $k \mapsto k^*_\alpha(x)$ is a homomorphism. Thus, by (3.4), we obtain a section $\tau^*_x(k) = k^*_\alpha(x)$ of $P^* \times_{G^*} [\text{Hom}(K, G^*)]$, and hence an action of K on P^* which commutes with the G^* -action. If further there are local gauges where the k_α are K -valued, then $K \cap N$ acts trivially so we get an action of $K^* = K/(K \cap N)$ which is a subgroup of G^* . Hence we get the situation studied in Section 2, with K^* acting on P^* .

Finally, if we restrict $k_\alpha(x)$ to be constant (and equal to k) in some gauge, then K^* acts rigidly on P^* . This can happen, as we have seen, if and only if P^* reduces to $Z_{G^*}(K^*)$, the centralizer of K^* in G^* . If

$$(3.5) \quad H' = \{g \in G \mid U(g)U(k) = U(k)U(g) \quad \forall k \in K\},$$

then this centralizer is $H^* = H'/N$. Since $P^* = P/N$, $P^*/H^* = P/H'$ and thus P^* reduces to H^* if and only if P reduces to H' . We summarize:

THEOREM 3.1

If K acts pointwise on a generic matter field Φ transforming under a unitary representation U of G on E so that there are local gauges (V_α, s_α) where this action is rigid,

$$(3.6) \quad (k \cdot \Phi)^\alpha(x) = U(k)\Phi^\alpha(x),$$

then P reduces to the subgroup (3.5). Conversely, any reduction of P to H' induces an action of K on Φ .

This is the case in particular when K acts internally on P . Indeed, H in (2.2) is a subgroup of H' . Alternatively, observe that if the action of K on P is associated to $\tau: P \rightarrow \text{Hom}(K, G)$, then

$$\tau^*_x(k) = (\tau_x(k))^*$$

defines an action of K (and thus of K^*) on P^* . Φ can be viewed alternatively as a section of $P^* \times_{G^*} E$, or as an equivariant function $P^* \rightarrow E$,

$$\Phi(p^* g^*) = U(g^{*-1}) \Phi(p^*), \quad g^* \in G^*, p^* \in P^*.$$

Observe that the action of K on Φ is deduced from that of K^* on P^* ,

$$(3.7) \quad (k \cdot \Phi)(p) = U(\tau_p^*(k)) \Phi(p),$$

(since $\Phi(p) = \Phi(p^*)$, whose local form is

$$(3.8) \quad (k \cdot \Phi)^\alpha(x) = U(\tau_x^*(k)) \Phi^\alpha(x).$$

The results of this section apply, besides monopoles (Section 7), to classical particles in external Yang-Mills fields [33].

4. INTERNAL SYMMETRIES

Let us now assume that our principal G -bundle P carries a connection form A , and let K be a subgroup of G acting on P internally. Let this action be given by τ . This allows us to define the action of a $k \in K$ on the Yang-Mills connection A , $(k \cdot A) = (k^{-1})^* A$, where $*$ denotes the pullback of a differential form. We shall call K an internal symmetry group for the Yang-Mills field A if this action preserves the connection,

$$(4.1) \quad (k \cdot A) = A.$$

cf. [7-16]. If K is a compact, connected Lie group with Lie algebra \hat{k} , any $k \in K$ can be written as $k = \exp \kappa$. (4.1) implies that the vectorfield

$$(4.2) \quad \tilde{\kappa}(p) = \left. \frac{d}{dt} \right|_{t=0} (\exp -t\kappa) \cdot p$$

is invariant under the right action of G on P , and

$$(4.3) \quad L_{\tilde{\kappa}} A = 0.$$

On the other hand, $\tilde{\kappa}(p) = \hat{\xi}_p$, the fundamental vectorfield at p associated to the infinitesimal right action of G at $p \in P$ for some $\xi \in \mathfrak{g}$. Denote $(\omega_\kappa(p)^\wedge)_p = \tilde{\kappa}(p)$. Alternatively, consider

$$(4.4) \quad \omega_\kappa(p) = \left. \frac{d}{dt} \right|_{t=0} \tau_p(\exp -t\kappa).$$

For each $p \in P$, the map $\hat{k} \ni \kappa \mapsto \omega_\kappa(p) \in \mathfrak{g}$ satisfies

$$(4.5) \quad \omega_{[\kappa_1, \kappa_2]}(p) = [\omega_{\kappa_1}(p), \omega_{\kappa_2}(p)] \text{ and } \omega_\kappa(p_0) = \kappa.$$

$\omega_\kappa(pg) = \text{Adg}^{-1} \omega_\kappa(p)$ for each κ . ω_κ is therefore an adjoint "Higgs - type" field.

To express (4.3) another way, observe that $L_{\tilde{\kappa}} A = dA(\tilde{\kappa}, \cdot) + d(A(\tilde{\kappa}))$ by the Cartan lemma. But $dA(\tilde{\kappa}, \cdot) = DA(\tilde{\kappa}, \cdot) = [A(\tilde{\kappa}), A(\cdot)] = [A(\cdot), A(\tilde{\kappa})]$ because $DA = F$ is horizontal. Finally, $A_p(\tilde{\kappa}_p) = A_p(\omega_\kappa(p)^\wedge) = \omega_\kappa(p)$, since $A(\hat{\xi}) = \xi$ for any

connection A on P. So $L_K^* A = D\omega_K$. This yields:

PROPOSITION 4.1

A connected subgroup K of G acting internally on P is an internal symmetry for the connection A if and only if the adjoint "Higgs" field ω_K associated to its infinitesimal action is covariantly constant for each x ,

$$(4.6) \quad D\omega_K = 0.$$

cf. [5,7,16]. When expressed in a local gauge $\{V_\alpha, s_\alpha\}$ (4.1) and (4.6) become [7-16]

$$(4.7) \quad A^\alpha(x) = \text{Ad } \tau_x^\alpha(k) A^\alpha(x) - d\tau_x^\alpha(k) [\tau_x^\alpha]^{-1}$$

and

$$D\omega_K^\alpha = d\omega_K^\alpha + [A^\alpha, \omega_K^\alpha] = 0$$

respectively, where $A^\alpha = s_\alpha^* A$, and

$$(4.8) \quad \omega_K^\alpha(x) = \frac{d}{dt} \bigg|_{t=0} \tau_x^\alpha(\exp(-tk)),$$

$\omega_K^\alpha(x)$ is just a local representative for ω_K , $\omega_K^\alpha(x) = \omega_K(s_\alpha(x))$, as anticipated by the notation.

Conversely, if we can find a bracket-preserving linear map $\kappa \rightarrow \omega_K$ satisfying (4.5) which associates covariantly constant adjoint Higgs-type fields ω_K to each $\kappa \in \hat{K}$, $(\exp(-\kappa) \cdot p = \exp(\omega_K(p))^\wedge)$ (exponential of a vectorfield) provides us with an internal action of $k = \exp(-\kappa) \in K$ on P. In fact, $\tau_p(\exp(-\kappa)) = \exp(\omega_K(p))$. (exponential in the group).

All solutions of (4.6) are found by parallel transport [18,7]. $\omega_K(p)$ belongs therefore, for all $p \in P$, to a single adjoint orbit of G. Hence we get

PROPOSITION 4.2

The action of an internal symmetry group K on P is rigid.

As we have seen in Section 2, to have a rigid internal action of K is equivalent to requiring that the bundle (P, G) reduces to $H=Z_G(H)$. In terms of ω this reduction is obtained

as $Q = \{p \in P \mid \omega_K(p) = \kappa\}$ [10]. This implies

PROPOSITION 4.3

The action of K on (P, A) defined by the (Q, H) (where $H = Z_G(K)$) is an internal symmetry if and only if the connection form A reduces to Q. This happens if and only if $H=Z_G(K)$ contains the holonomy group of A. In this case Q contains the holonomy bundle and so is unique.

In particular, the full gauge group G is a group of internal symmetries if and only if the connection reduces to the $Z(G)$ -bundle which characterizes the left action of G on P. This is equivalent to requiring that the generators of the holonomy group lie in the centre of [23].

Similarly, let us consider a matter field Φ , and assume there is an internal action of K on Φ determined by a section τ^* of $P^* \times_G^* [\text{Hom}(K^*, G^*)]$. We shall say that this action of K is an internal symmetry for the matter field Φ , if

$$(4.9) \quad (k \cdot \Phi)(p) = U(\tau_p^*(k))\Phi(p) = \Phi(p).$$

In a local (in particular in a rigid) gauge this reads

$$(4.10) \quad U(\tau_x^*(k))\Phi^\alpha(x) = \Phi^\alpha(x) \text{ and } U(k) \cdot \Phi^\alpha(x) = \Phi^\alpha(x)$$

respectively. We can work also infinitesimally:

$$(4.11) \quad \tilde{\kappa}^*(p^*) = \frac{d}{dt} \bigg|_{t=0} (\exp(-tk))^* \cdot p^*$$

and

$$\omega_{\kappa^*}^*(p^*) = \frac{d}{dt} \bigg|_{t=0} \tau_p^*(\exp(-tk))^*$$

provide us with a vertical vectorfield κ^* and an adjoint "Higgs" field $\omega_{\kappa^*}^*$ on $P^*=P/N$. The infinitesimal action of \hat{K}^* (the Lie algebra of K^*) on Φ reads

$$(4.12) \quad (\tilde{\kappa}^* \cdot \Phi)(p) = L_{\tilde{\kappa}^*} \Phi(p) = \omega_{\kappa^*}^*(p) \cdot \Phi(p)$$

where the dot \cdot denotes the action of the Lie algebra induced by U. The definition of a symmetric matter field

reads hence infinitesimally

$$(4.13) \quad \omega^*_{\kappa^*} \cdot \Phi(p) = 0.$$

or in a local gauge

$$(4.14) \quad \omega^*_{\kappa^*} \alpha(x) \cdot \Phi^{\alpha}(x) = 0.$$

Let us now assume that we have a Yang-Mills potential A and a covariantly constant Higgs field Φ and that we are interested in their simultaneous symmetries. First, K implementable on P implies that K is implementable also on $P \times_G E$. Furthermore, $\omega^*_{\kappa}(x) = (\omega_{\kappa}(p))^*$. In particular if N is discrete, $\mathcal{K}^* = \mathcal{K}$ and $\mathcal{G}^* = \mathcal{G}$, so the star can be dropped. Both ω_{κ} and Φ are now found by parallel transport, $\Phi^{\alpha}(x) = \text{Adg}^{\alpha}(x)\Phi_0$ and $\omega_{\kappa}^{\alpha}(x) = \text{Adg}^{\alpha}(x)\kappa$, where g^{α} is the non-integrable phase factor. (4.13) reduces hence to

$$(4.15) \quad \kappa \cdot \Phi_0 = 0.$$

PROPOSITION 4.4

K is a symmetry group for a covariantly constant matter field Φ if and only if K belongs to the little group of a basepoint Φ_0 from the orbit where Φ takes its values.

As a first illustration, consider the non-Abelian Bohm-Aharonov experiment [7, 33-35] proposed by Wu and Yang to test the existence of gauge fields. Here one considers a principal G -bundle P over the punctured plane $M = \mathbb{R}^2 \setminus \{0\}$ endowed with a flat connection A . Such bundles are classified by classes $[P]$ in $\pi_0(G)$. The homotopy exact sequence

$$(4.16) \quad \dots \rightarrow \pi_1(G) \xrightarrow{\text{Ad}_*} \pi_1(G/Z(G)) \xrightarrow{\delta} \pi_0(Z(G)) \xrightarrow{i_*} \pi_0(G) \rightarrow \dots$$

$[Q] \qquad [P]$

analogous to (6.3) and induced by $Z(G) \rightarrow G \rightarrow G/Z(G)$ shows that G is implementable iff $[P]$ belongs to $\text{Im } i_*$. If this condition is satisfied, there may be still an ambiguity: the different implementations - the inequivalent reductions to $Z(G)$ - are classified by $\text{Ker } i_*$. For $G = \text{SU}(2)$ for example, P is trivial because $\text{SU}(2)$ is connected.

$\pi_1(\text{SU}(2)) = 0$, $Z(G) = \mathbb{Z}_2$, $G/Z(G) = \text{SO}(3)$ so there exist two gauge-inequivalent implementations corresponding to the reductions of P to a trivial or to a twisted bundle respectively [7].

When do we get an internal symmetry? Let us consider more generally a principal G -bundle P over a connected manifold M carrying a flat connection A .

The horizontal distribution of A is integrable by the Frobenius theorem. Let us choose a reference point p_0 in P , and denote by Q the leaf of the horizontal distribution through p_0 . Q is a covering of M which is a reduction of P with a discrete subgroup Γ of G as structure group. As a matter of fact, Γ is just the holonomy group of A at p_0 . According to Prop. 4.3 G acts as a symmetry for (P, A) iff Γ is in $Z(G)$.

In the non-Abelian Bohm-Aharonov experiment Γ consists of powers of Φ , the non-integrable phase factor calculated along a loop which winds once around the origin [33-35]. Consequently, $\text{SU}(2)$ acts as a group of internal symmetries only for $\Phi = 1$ (for the first implementation) or for $\Phi = -1$ (for the second implementation) [7]. The physical consequences are explained in [35].

5. MONOPOLE BUNDLES

The asymptotic properties of monopoles are determined by a principal G -bundle P over the S^2 at infinity, where G , the residual symmetry group, is compact and connected. In Grand Unified Theories one starts in general with a trivial "unifying" bundle $\tilde{P} = \mathbb{R}^3 \times \tilde{G}$, where \tilde{G} - a compact and connected Lie group - is the "unifying group". At large distances the \tilde{G} -symmetry is spontaneously broken to a subgroup G of \tilde{G} . Geometrically, this means that over S^2 (\tilde{P}, \tilde{G}) is reduced to a principal G -bundle P . Any such reduction is produced by an equivariant "reducing map" [10]. Choosing a global trivialisation of \tilde{P} , the reducing map can be identified with a map $\Phi: S^2 \rightarrow \tilde{G}/G$ - the physical Higgs field. Φ defines a homotopy class $[\Phi] \in \pi_2(\tilde{G}/G)$, and the homotopy class $[P]$ is $\delta[\Phi]$, where $\delta: \pi_2(\tilde{G}/G) \rightarrow \pi_1(G)$ is the connecting homomorphism. δ is an isomorphism if \tilde{G} is simply connected. Both $[P]$ and $[\Phi]$ will be referred to as the Higgs charge in the sequel.

$$\pi_1(G) = \pi_1(G)_{\text{free}} + \pi_1(G_{\text{ss}}),$$

where $\pi_1(G)_{\text{free}} \cong \mathbb{Z}^p$, p is the dimension of $Z(\mathfrak{g})$, and G_{ss} is the semisimple subgroup of G generated by $[\mathfrak{g}, \mathfrak{g}]$. $\pi_1(G_{\text{ss}})$ is a finite Abelian group. The free part - which plays a particularly important role - is described as follows [23]: denote by $\Gamma = \{\xi \in \mathfrak{g} \mid \exp 2\pi\xi = 1\}$ and let $z: \mathfrak{g} \rightarrow Z(\mathfrak{g})$ be the projection of the Lie algebra of G onto its centre. $z(\Gamma)$, the image of Γ under z , is a lattice whose dimension is the same as that of $Z(\mathfrak{g})$. In [23] we proved the following theorem: Define, for any loop γ in G ,

$$(5.1) \quad \rho(\gamma) = \frac{1}{2\pi} \int_{\gamma} z(\theta) \in Z(\mathfrak{g}),$$

where $\theta = g^{-1}dg$ is the canonical (Maurer-Cartan) 1-form of G . ρ defines an isomorphism of $\pi_1(G)_{\text{free}}$ with $z(\Gamma)$.

Any loop in G is known to be homotopic to one of the form $\gamma(t) = \exp 2\pi\xi t$. For this γ $\rho(\gamma) = z(\xi)$. If ζ_1, \dots, ζ_p is a \mathbb{Z} -basis for $z(\Gamma)$, then $\rho(\gamma) = \sum m_i \zeta_i$ provides us with p "quantum" numbers m_1, \dots, m_p .

In [23] we gave also a second characterization of ρ , namely that $\rho(\Phi) = \rho(\delta[\Phi])$ can also be calculated as the integral of a 2-form over the 2-sphere at infinity,

$$(5.2) \quad \rho(\Phi) = \frac{1}{2\pi} \int_{S^2} \Phi^* \Omega,$$

where Ω is the projection to \tilde{G}/G of the $Z(\mathfrak{g})$ -valued 2-form $z(d\tilde{\theta})$ on \tilde{G} .

Here we give third construction, adapted from Chern-Weil theory [10], Vol. II.: let us consider an arbitrary connection form A on P , and denote by $F = DA$ its curvature form. $z(F)$ is a $Z(\mathfrak{g})$ -valued 2-form on P , which is horizontal and basic, since

$$r_g^* z(F) = z(r_g^* F) = z(\text{Ad} g^{-1} F) = z(F),$$

so $z(F)$ descends to S^2 to a $Z(\mathfrak{g})$ -valued 2-form Ω^A , $z(F) = \pi^* \Omega^A$. This two-form is closed,

$$d(z(F)) = z(dF) = z(DF - [A, F]) = z(DF) = 0,$$

since z vanishes on the derived algebra and $DF = 0$ by the Bianchi identity. The class $[z(F)]$ is known [10] to be independent of the choice of connection. This proves:

PROPOSITION 5.1

The cohomology class $[\Omega^A] \in H^2_{\text{dR}}(S^2) \times Z(\mathfrak{g})$ is independent of the choice of the connection A on P . Consequently,

$$(5.3) \quad \rho(P) = \frac{1}{2\pi} \int_{S^2} z(F) \in Z(\mathfrak{g})$$

depends only on the bundle P .

Let us now assume that (P, G) is the reduction of the trivial unifying bundle (\tilde{P}, \tilde{G}) defined by $\Phi: S^2 \rightarrow \tilde{G}/G$ and so can be identified with the pullback by Φ of \tilde{G} , viewed as a principal G -bundle over the orbit \tilde{G}/G [10]. The \mathfrak{g} -component of the Maurer-Cartan 1-form $\tilde{\theta}$ defines a connection on the principal G -bundle \tilde{G} whose pullback by Φ is a connection form A on P such that $z(DA) = \Phi^* \Omega$, where Ω is the 2-form defined above. Thus we have established the equivalence of our new construction with those given before.

Monopole fields must also satisfy the Yang-Mills-Higgs equations. Assuming a sufficiently rapid fall-off at infinity, the Yang-Mills-Higgs equation on S^2 reduces to $D^*F = 0$. The solution has been found by Goddard, Nuyts and Olive [17-20, 28-29]): Let us assume that P is a non-trivial

G-bundle over S^2 , carrying a connection form A which satisfies the YM equation $D^*F = 0$. Then there is a vector Π in generating a homomorphism $U(1) \rightarrow G$ such that P is associated to the Hopf bundle over S^2 and the field is $F = \nabla \Pi$ with ∇ the area form on the 2-sphere. Π is quantized, $\exp i\pi \Pi = 1$. The vector Π can be chosen without loss of generality in any given Cartan subalgebra of \mathfrak{g} . The transition function h of a monopole is thus homotopic to $h(t) = \exp i\pi \Pi t$, $0 \leq t \leq 1$, so $\rho(P)$ is simply

$$(5.4) \quad \rho(P) = z(2\Pi).$$

This theorem can also be reformulated by saying that the holonomy group of asymptotic monopole bundles is a $U(1)$, generated by the "non-Abelian charge" vector Π [29]. Conversely, given Π we are able to construct an asymptotic monopole configuration, see the following section.

6. REDUCTION OF MONOPOLE BUNDLES

THEOREM 6.1

The monopole bundle (P, G) is reducible to an H -bundle Q (where H is a closed subgroup of G) iff

$$(6.1) \quad [\Phi] \in \text{Im } \sigma_*$$

where $\sigma_*: \pi_2(\tilde{G}/H) \rightarrow \pi_2(\tilde{G}/G)$ is induced by the natural projection $\sigma: \tilde{G}/H \rightarrow \tilde{G}/G$. Equivalently, iff

$$(6.2) \quad [P] = \delta[\Phi] \in \text{Im } i_*$$

where $i_*: \pi_1(H) \rightarrow \pi_1(G)$ is induced by the inclusion $i: H \hookrightarrow G$. The inequivalent reductions from G to H are parametrized by the elements of $\pi_2(G/H)$.

Proof: Suppose first that there exists a reduction (Q, H) of (P, G) . (Q, H) is a reduction of (\tilde{P}, \tilde{G}) also, and is thus determined by a reducing map ("Higgs field") $\Psi: S^2 \rightarrow \tilde{G}/H$. This reduces P if and only if each H -coset is contained in a corresponding G -coset. That is, if and only if $\Phi = \sigma(\Psi)$. But this implies $[\Phi] = \sigma_*[\Psi]$.

Conversely, if $[\Phi] \in \text{Im } \sigma_*$, then $[\Phi] = \sigma_*[\Psi_1]$ for some $\Psi_1: S^2 \rightarrow \tilde{G}/H$. Φ and $\sigma(\Psi_1)$ are hence homotopic and thus gauge-equivalent [8, 9, 10], so there exists a map $g(x)$ such that $\Phi(x) = g(x) \cdot \sigma(\Psi_1(x))$. But σ is a G -map, so putting $\Psi(x) = g(x) \cdot \Psi_1(x)$, $\Phi(x) = \sigma(\Psi(x))$, and hence we get a reduction Q of P defined by Ψ . Consider

$$(6.3) \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & \pi_2(G/H) & & & \\ & & \delta \downarrow \delta & & (j \circ i)_* & & \\ 0 \rightarrow & \pi_2(\tilde{G}/H) & \rightarrow & \pi_1(H) & \rightarrow & \pi_1(\tilde{G}) & \rightarrow \dots \\ & \downarrow \sigma_* & & \downarrow i_* & & \parallel & \\ 0 \rightarrow & \pi_2(\tilde{G}/G) & \xrightarrow{\delta} & \pi_1(G) & \xrightarrow{j_*} & \pi_1(\tilde{G}) & \rightarrow \dots \\ & & & \downarrow & & & \end{array}$$

It follows from this diagram, that $[\Phi] \in \text{Im } \sigma_*$ if and only if

(6.2) holds.

Finally, by Theorem 2.5 two reductions (Q_1, H) and (Q_2, H) of (P, G) are equivalent iff $[Q_1] = [Q_2] \in \pi_1(H)$. The inequivalent reductions are hence labelled by $\text{Ker } i_*$ which is, according to the diagram, just $\pi_2(G/H)$.

It follows from (6.2) that if $\text{Im } i_* = 0$, (in particular when H is simply connected), then (P, G) reduces to a subbundle Q with structure group H if and only if the Higgs charge of Φ is zero and so the G -bundle P is trivial [25, 30].

PROPOSITION 6.2

The structure group G of P can be reduced to H if and only if h_P is homotopic to a loop in H . In this case

$$(6.4) \quad \rho_G([P]) = z_G(\rho_H([Q])).$$

If $\pi_1(G)$ is free and H is Abelian, then (6.4) is also sufficient.

Indeed, the reductions (P, G) and (Q, H) of (\tilde{P}, \tilde{G}) are compatible if and only if the transition functions are homotopic in G , $[\exp i\pi\Pi_{Pt}] = [\exp i\pi\Pi_{Qt}] \in \pi_1(G)$. Next, the projection maps $z_G: \mathcal{G} \rightarrow Z(\mathcal{G})$ and $z_H: \mathcal{H} \rightarrow Z(\mathcal{H})$ satisfy $z_G(z_H(\eta)) = z_G(\eta)$, $\eta \in \mathcal{H} \hookrightarrow \mathcal{G}$. This follows from $z_G([\mathcal{G}, \mathcal{G}]) = 0$, observing that $\eta = z_H(\eta) + \eta'$, where $\eta' \in [\mathcal{H}, \mathcal{H}] \subset [\mathcal{G}, \mathcal{G}]$. Notice that any H -connection on Q extends naturally to a G -connection on P . Let F denote the curvature. On Q $z_G F = z_G(z_H F)$, since F is H -valued. (6.4) follows then from the definition.

Finally, let $h_P(t) = \exp 2\pi\xi t$ and $h_Q(t) = \exp 2\pi\eta t$ be the transition functions of the bundles P and Q . Then $\rho_G([P]) = z_G(\xi)$ and $\rho_H([Q]) = \eta$. Thus $\rho_G(h_Q) = z_G(\eta) = z_G(\xi)$. Therefore, if $\pi_1(G)$ is free and H is Abelian, h_P and h_Q are homotopic loops in G since ρ_G is now an isomorphism.

An interesting insight is gained by proceeding backwards. Let us start with a (right) principal H -bundle Q over the two-sphere, and assume $i: H \hookrightarrow G$ is a subgroup of G . $(q, g) \sim (gh, h^{-1}g)$ is an equivalence relation on $Q \times G$ and the set of equivalence classes (denoted by $\{q, g\}$ here) yields

the associated bundle $PQ = Q \times_H G$. PQ is a principal G -bundle with G -action $\{q, g\} \rightarrow \{q, gg'\}$, $g' \in G$. (Q, H) is furthermore a reduction of (PQ, G) , $Q \simeq (\{q, e\} | q \in Q)$.

If (Q, H) has isomorphism class $[Q] \in \pi_1(H)$, the class of PQ is $[PQ] = i_*[Q] \in \pi_1(G)$. PQ is thus isomorphic to a given G -bundle P iff

$$(6.5) \quad i_*[Q] = [P].$$

If (6.5) is satisfied, (Q, H) is a reduction of (P, G) by construction. This shows also that the different reductions of P are parametrized by $\text{Ker } i_* = \pi_2(G/H)$.

This procedure allows us also to construct asymptotic monopole configurations [28]. The non-Abelian charge vector Π is written as $\Pi = (n/2)\xi$, n an integer and ξ a minimal $U(1)$ generator, because Π is quantized. Denote by Y^n the Hopf-bundle S^3/Z_n . $H = \{\exp 2\pi i t \xi | 0 \leq t \leq 1\}$ is a $U(1)$ subgroup of G . Y^n can be viewed also as a principal H -bundle with H -action $\gamma \rightarrow \gamma h = h(e^{2\pi i t})$ for $h = \exp 2\pi i t \xi$. The associated bundle $P(\Pi) = Y^n \times_H G$ is a principal G -bundle having transition function $h(\theta) = \exp 2i\theta\Pi$, $0 \leq \theta \leq 2\pi$ being the angle parametrizing the equatorial circle of S^2 . The natural connection $A^{(n)} = n\bar{y}dy/i$ of Y^n extends to a connection $A(\Pi)$ on $P(\Pi)$; as a matter of fact, $\gamma(\Pi) = (\{y, e\} | y \in Y^n)$ is the holonomy bundle of $(P(\Pi), A(\Pi))$. This latter is an asymptotic monopole bundle iff $[P(\Pi)] \in \text{Ker } j_*$ for $j: G \hookrightarrow \tilde{G}$ [20, 28]. Under suitable conditions such asymptotic solutions can be extended to the interior region [31, 32].

The connection on (P, G) determined by the non-Abelian charge vector reduces to a subbundle (Q, H) iff this latter contains the holonomy bundle. We conclude:

THEOREM 6.3

The Yang-Mills connection A of a principal G -bundle P over S^2 defined by the non-Abelian charge Π reduces to a subbundle (Q, H) if and only if $\Pi \in \mathcal{H}$.

For example, (P, G) reduces to a $U(1)$ subgroup $H = \{\exp 2\pi i t \eta | 0 \leq t \leq 1\}$, where η is a minimal generator, iff $[\exp 2\pi i t \eta] =$

$[\exp i\pi t\Pi] \in \pi_1(G)$. A necessary condition for this is $z(\eta) = 2z(\Pi)$. This is also sufficient if $\pi_1(G)$ has no finite part. The YM connection A reduces also iff the reduced H -bundle Q contains $Y(\Pi)$ which happens iff $2\Pi = n\eta$ for a suitable integer n .

These results provide us with topological informations concerning the "fate" of monopoles under successive symmetry breakings $G \rightarrow H$ [24-27]. The topological condition found in [24,25] for its survival means exactly that the G -bundle P reduces to an H -bundle Q . On the other hand, the second condition given in [25-27] requires that the Yang-Mills connection reduces also.

7. THE COLOR PROBLEM IN MONOPOLE THEORY

Let us consider a Grand Unified monopole (A_j, Φ) with "residual" symmetry group G . G is the little group of a basepoint Φ_0 in the orbit where the Higgs field takes asymptotically its values. Our previous results imply

THEOREM 7.1

A subgroup $K \subset G$ is (rigidly) implementable if and only if, over S^2 , the monopole bundle (P, G) reduces to a $Z_G(K)$ -bundle Q . The rigid actions of K are in (1-1) correspondence with reductions to $Z_G(K)$. The necessary and sufficient condition of implementability of K is

$$(7.1) \quad \delta[\Phi] \in \text{Im } i_*,$$

where i_* is the homomorphism between homotopy groups induced by the inclusion map $i: Z_G(K) \hookrightarrow G$. The inequivalent reductions are in (1-1) correspondence with the elements of $\pi_2(G/Z_G(K))$.

An alternative proof is obtained using the inverse technique of the previous section. Denote in fact $Z_G(K)$ by H and consider an H -bundle Q over S^2 . Such bundles exist for each element in $\pi_1(H)$. Form the associated bundle $P(Q)$. $p = qg$ with $q \in Q$ and $g \in G$ for each $p \in P(Q)$, because Q is a reduction of $P(Q)$. Set

$$(7.2) \quad k \cdot p = pg^{-1}kg.$$

If $p = q'g'$ with $q' \in Q$ and $g' \in G$, then $q' = qh$, $g' = h^{-1}g$ for some $h \in H$. Thus

$$p(g')^{-1}kg' = pg^{-1}hkh^{-1}g = pg^{-1}kg,$$

because h is in $Z_G(K)$. (7.2) is thus a well-defined action of K on $P(Q)$. Furthermore,

$$(7.3) \quad \tau_p = \text{Ad } g^{-1}i,$$

showing that this implementation is rigid. Notice also that $\tau|_Q = i$.

As we have seen in Section 6, $P(Q)$ is isomorphic to a

given monopole bundle (P, G) iff (6.5) is satisfied, and the different possibilities are parametrized by $\pi_2(G/H) = \pi_2(G/Z_G(K))$.

For $K = G$ we have some more results: [1,2]:

PROPOSITION 7.2

G is implementable if and only if the transition function $h(t) = \exp 4\pi t\Pi$, $0 \leq t \leq 1$ of the bundle P is homotopic to a loop in the centre of G . This happens iff

$$(7.4) \quad \delta[\Phi] \in \pi_1(G)_{\text{free}}$$

and

$$(7.5) \quad \exp 2\pi \rho_G(P) = \exp 4\pi z(\Pi) = 1.$$

The implementation of G is unique.

Indeed, for $i : Z(G) \rightarrow G$ i_* belongs to $\pi_1(G)_{\text{free}}$. On the other hand, a $Z(G)$ bundle Q is represented by a loop $\exp 2\pi \zeta t$, $0 \leq t \leq 1$, where ζ is in $Z(\mathfrak{g})$. Then $\rho_H(Q) = z_H(\zeta) = \zeta$. By $\rho_G(P) = \zeta$ by 6.4. However, $\rho_G(P) = z(2\Pi)$.

Conversely, (7.4) and (7.5) are also sufficient: $\gamma(t) = \exp 2\pi \rho_G(P)t$ is now a loop in $\text{Im } i_*$ whose homotopy class is $\delta[\Phi]$, because $\rho_G(\gamma) = \rho_G(P)$ and ρ_G , when restricted to the free part, is an isomorphism. G admits at most one implementation, since $G/Z(G)$ is a Lie group and has thus trivial second homotopy. So i_* is now injective.

To express this result another way, decompose the non-Abelian charge vector as $\Pi = z(\Pi) + \Pi'$, where Π' belongs to $[\mathfrak{g}, \mathfrak{g}]$. Denote by G_{ss} the semisimple subgroup of G whose Lie algebra is $[\mathfrak{g}, \mathfrak{g}]$, and let G_{ss}^* be the simply connected covering group of G_{ss} .

PROPOSITION 7.3

G is implementable if and only if $\exp 4\pi t\Pi'$, $0 \leq t \leq 1$ is a contractible loop in G_{ss} . This happens iff

$$(7.6) \quad \exp^* 4\pi \Pi' = 1,$$

where \exp^* is the exponential map in G_{ss}^* .

Proof: $z(\Pi)$ commutes with everything, and thus

$$\exp 4\pi \Pi' = (\exp 4\pi \Pi) \cdot (\exp 4\pi z(\Pi))^{-1} = (\exp 4\pi z(\Pi))^{-1}.$$

(7.5) and (7.6) are thus equivalent; in particular, $\exp 4\pi t\Pi'$, $0 \leq t \leq 1$ is a loop in G_{ss} . $[P]$ is now decomposed as

$$[P] = [\exp 4\pi t z(\Pi)] + [\exp 4\pi t \Pi'] \in \pi_1(G)_{\text{free}} + \pi_1(G_{\text{ss}}),$$

and hence (7.4) is equivalent to $\exp 4\pi t \Pi'$ contractible in G_{ss} . But $\pi_1(G_{\text{ss}})$ is known to be Γ/Γ^* , where Γ (respectively Γ^*) are the unit lattices of G_{ss} (respectively of G_{ss}^*). Thus $\exp 4\pi t \Pi'$ contractible means exactly (7.5).

This is seen alternatively by noting [1,7] that, according to the diagram (6.3), $[P] \in \text{Im } i_*$ exactly when $\text{Ad}^*[P] = 0$, i.e. the transition function $\text{Ad}h$ is contractible in $(\text{Aut } G)_0 = \text{Int } G = G/Z(G)$. But the condition for this is just (7.6), since $G/Z(G)$ has $[\mathfrak{g}, \mathfrak{g}]$ for Lie algebra.

(7.5) can be translated into numbers: let ζ_1, \dots, ζ_p be a \mathbb{Z} -basis for $z(\Gamma)$ (assumed non-empty), then $\rho(P) = \sum m_j \zeta_j$. On the other hand, there exist least positive integers such that $m_j \zeta_j \in \Gamma$ [23]. Thus (7.5) can hold only if, for each j , m_j/M_j is an integer, say n_j . Consequently

PROPOSITION 7.4

G is implementable if and only if (7.4) is valid and

$$(7.8) \quad m_j = n_j M_j \text{ for suitable integers } n_j.$$

The case $K \neq G$ is similar but more complicated, cf. [7].

Next, Prop. 4.4 and Theorem 6.3 imply

THEOREM 7.5

$K \subset G$ is an internal symmetry group if and only if the loop $h_P(t) = \exp 4\pi t\Pi$ lies in $Z_G(K)$. This happens iff

$$(7.9) \quad \text{Ad } k \Pi = \Pi, \quad \forall k \in K.$$

In particular, G is an internal symmetry iff Π lies in the centre. The action is then unique.

Indeed, the holonomy group of a monopole-bundle is generated by the non-abelian charge Π and (7.9) means exactly that $K \subset Z_G(\Pi)$. $[\zeta, \Phi_0] = 0$ is automatically satisfied, since $\zeta \in \mathfrak{g}$ stabilizes Φ_0 . Alternatively, the implementation defined by a reduction (Q, H) is a symmetry iff (Q, H) contains the holonomy bundle $Y(\Pi)$.

As an illustration, consider a GUT with residual group $G = SO(3)$. Such a situation arises, e.g., when $G = SU(3)$ is broken by a Higgs $\underline{6}$ [18, 27]. Choose in $\mathfrak{so}(3)$ the Cartan algebra

$$(7.10) \quad \mathcal{Y} = aL_3 = \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a \in \mathbb{R}.$$

The non-Abelian charge vector can be gauge-rotated into \mathcal{Y} . Then $\Pi = (m/2)L_3$ where m is an integer. $\pi_1(SO(3)) = \mathbb{Z}_2$, $[\exp 2\pi m L_3] = m$ (modulo 2). Topologically non-trivial solutions arise therefore if m is odd. Denote by P the corresponding $SO(3)$ -bundle.

$G = SO(3)$ is not implementable on P : (7.6) would require, in fact, m to be even.

Consider now a $U(1)$ subgroup K with minimal generator ξ , $K = \{\exp 2\pi i t \xi\}$. ξ is conjugate to L_3 , and hence $\pi_1(SO(3)) [\exp 2\pi i n t \xi] = [\exp 2\pi i n L_3] = n$ (modulo 2). $Z_{SO(3)}(K) = K$, so the interesting part of Diagramm (6.3) becomes

$$(7.11) \quad \begin{array}{ccc} \longrightarrow & \pi_1(K) & \xrightarrow{i_*} \pi_1(SO(3)) \longrightarrow \\ & \parallel & \parallel \\ & \mathbb{Z} & \longrightarrow \mathbb{Z}_2 \\ & n & \longrightarrow n \text{ (modulo 2)} \end{array}$$

Theorems 7.1 and 7.5 tell us therefore that

- (i) K is implementable on P iff n is odd;
- (ii) For $n = 2k+1$ $\text{Ker } i_* \subset \mathbb{Z}$: there is a different implementation for each k , corresponding to the different reductions to K ;
- (iii) K is a symmetry iff Π and ξ are parallel, $\Pi = (n/2)\xi$ for some integer n .

We are able to construct the bundles explicitly:

choose an integer n and consider the Hopf bundle $Y^n = S^3/\mathbb{Z}_n$ where S^3 is viewed as sitting in \mathbb{C}^2 . Y^n is a two-sided $U(1)$ bundle with actions $z: y \rightarrow z \cdot y = (zy_1, zy_2)$ and $z': y \rightarrow y \cdot z' = (y_1 z', y_2 z')$, $y = (y_1, y_2) \in \mathbb{C}^2$, $z, z' \in U(1)$. Y^n can be viewed alternatively as a two-sided principal K -bundle with $k = \exp 2\pi i a$ acting as $k \cdot y = (e^{2\pi i a} y)$ and $y \cdot k = y(e^{2\pi i a})$ respectively.

The associated bundle $p(n) = Y^n \times_K SO(3)$, is a right principal $SO(3)$ bundle. Y^n is identified with $Y^\xi = \{[y, e] \mid y \in Y^n\}$ and so is a reduction of $p(n)$.

The right action of K on Y^n was used to construct $p(n)$. However, we still have a left action of K on Y^n which extends to a left action of K on $p(n)$ according to

$$(7.12) \quad k[y, g] = [k \cdot y, g] = [y, k^{-1}g] = [y, g] \cdot \text{Adg}^{-1}k,$$

where $k = \exp 2\pi i a$. Hence, for $p = [y, g]$,

$$(7.13) \quad \tau_p(k) = \text{Adg}^{-1}k,$$

The transition function of the principal K -bundle $Y(\xi)$ is $h(\theta) = \exp \theta \xi n$. Hence $\pi_1(SO(3)) [P_n] = n$ (modulo 2): $p(n)$ is the trivial bundle for n even and is isomorphic to P for n odd. Our construction provides us hence with a rigid action of K on P for each odd integer n , as expected. These actions are obviously inequivalent.

The action of K as constructed above is a symmetry for the monopole field A given by the non-Abelian charge vector Π iff $Y(\xi)$ contains the holonomy bundle, which happens iff $\Pi = (n/2)\xi$.

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